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On the origin of the circular hydraulic jump: a differential analysis

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Abstract

It was believed that commonly seen kitchen sink circular hydraulic jumps are created due to gravity [1]. In a recent paper, we disputed this view and through a series of experiments and a theoretical analysis, we proved that gravity does not play any significant role in the formation of a kitchen sink scale circular hydraulic jump [2]. For the theoretical explanation, we performed a control volume analysis [2]. Here, we have obtain the same result by performing a differential analysis based on the energy equation. The energy equation in differential form, which represents a differential element in the bulk fluid, does not contain the surface energy term, which is only introduced when the equation was integrated from bottom to the free surface of the fluid. The conclusions remain the same; in thin film flow, the downstream transport of surface energy is the important term determining the location of the hydraulic jump and gravity does not play any significant role. The kitchen sink scale circular hydraulic jump are created due to surface tension where capillary waves play the role of gravity waves in a traditional jump and delineate the transition from the supercritical to subcritical flow in the liquid film, related to these jumps.

Introduction

When a jet of water falls vertically from a tap on to the base of a domestic sink, the water spreads radially outwards in a thin film until it reaches a radius where the film thickness increases abruptly (see figure 1a). This abrupt change in depth is the circular hydraulic jump. A similar phenomenon is observed on vertical and inclined surfaces, where the liquid film spreads radially outwards before forming a jump. All existing theories invoke gravity in the origin of the hydraulic jump [4, 3] implying that the jump location should be sensitive to the orientation of the surface. However, we observed that, under the same flow conditions, normal impingement of a liquid jet gives a circular hydraulic jump with the same initial radius irrespective of the orientation of the surface. On a horizontal surface, the jump (figure 1a) stays approximately at the same location until the liquid reaches the edge of the plate, which changes the downstream flow and the subsequent position of the jump. On a vertical plate, where the spreading liquid film and gravity are coplanar, an approximately circular hydraulic jump is formed initially (see figure 1b). The thick liquid film beyond the hydraulic jump then drains downwards due to gravity [5, 6, 7, 8, 9]. Similarly, when a vertical liquid jet impinges onto a ceiling, a circular hydraulic jump is formed (see figure 1c). Under the influence of gravity, the thick liquid film beyond the hydraulic jump falls as droplets or as a continuous film forming a water bell [10].

Figure 1, which is reproduced from Bhagat et al.(2018) shows that in all three cases the hydraulic jump has almost the same radius ($R \approx 26\,\mathrm{mm}$). These three experiments show unequivocally that gravity plays no role in the formation of the circular hydraulic jump in a thin liquid film and that gravity only affects the jump after it is formed.

Watson (1964) proposed the first description of a thin-film circular hydraulic jump (such as in a sink), incorporating the viscous friction in the thin liquid film and balancing the momentum and hydrostatic pressure across the jump [3]. Watson's solution, which involves gravity, is not a predictive theory since to apply the force balance, it requires experimental measurement of the liquid film thickness downstream of the jump. Moreover, for smaller flow rates, Watson's balance overestimates the jump radius by as much as 50% [11]. Bush & Aristoff (2003) incorporated the effect of surface tension in Watsons theory but argued that its influence was small as its effect was confined to the hoop stress associated with the increase in circumference of the jump and concluded that surface tension is important only when jump radii are small. They wrote *"Our study demonstrates that surface tension becomes dynamically significant when the radial curvature force becomes comparable with the hydrostatic pressure forces, that is, when 2/Bo becomes appreciable. While the influence of surface tension is generally weak in terrestrial experiments, it becomes appreciable for jumps of small radius and height. Moreover, its influence will be heightened dramatically in a microgravity setting, or when internal jumps arise between immiscible fluids of comparable density"* [11].

Bohr et al.(1993) connected the inner and the outer solutions for radial flow through a shock in shallow water and obtained a scaling relation $R \sim Q^{5/8}\nu^{-3/8}g^{-1/8}$ where R, Q, ν, and g are the jump radius, the jet volume flux, the fluid kinematic viscosity and gravitational acceleration, respectively [4].

The experimental observations presented by Bhagat et al.(2018) show a sharp departure from these previous approaches [2]. Furthermore, the previous theories require information or feedback from the liquid film downstream of the hydraulic jump to predict its location [4, 3], but the initially spreading liquid film does not receive information of this nature. Accounting for radial spread, momentum, viscous force and surface tension Bhagat et al.(2018) gave a scaling relationship $R \sim Q^{3/4}\rho^{1/4}\nu^{-1/4}\gamma^{-1/4}$ and concluded that the thin film hydraulic jumps are created due to surface tension and gravity does not play any significant role. To obtain a theoretical description, Bhagat et al.(2018) applied a flux average energy balance. Here, we solve the mechanical energy equation in differential form and obtained a similar result.

Theory

We consider cylindrical coordinates r and z, the radial and jet-axial coordinates, respectively, u and w the associated velocity components (figure 2), and assume circular symmetry about the jet axis. In the boundary layer approximation, the equations are governing steady flow in a thin film are

$$\frac{\partial(ru)}{\partial r}+\frac{\partial(rw)}{\partial z}=0, \tag{1}$$

$$u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}=\nu\frac{\partial^2 u}{\partial z^2}. \tag{2}$$

The no slip boundary conditions at the substrate and zero shear stress condition at free surface are

$$u=w=0, \qquad at \quad z=0, \tag{3}$$

$$\frac{\partial u}{\partial r}=0, \qquad at \quad z=h. \tag{4}$$

For constant jet flow rate Q the radial velocity satisfies $2\pi r\int_0^h u\,dz=Q$

In order to analyse the jump we use the *ansatz* developed by [3] for the velocity within the thin film. We write the radial velocity as $u=u_s f(\eta)$, $\eta\equiv z/h(r)$ $(0\le\eta\le 1)$, where η is the dimensionless thickness of the film and u_s is the velocity at the free surface. Using continuity we define the flux-average velocity $\bar{u}\equiv C_1 u_s$ by

$$\int_0^h ur dz=u_s rh\int_0^h f(\eta)d\eta=C_1 u_s rh\equiv\bar{u}rh=\frac{Q}{2\pi}=\text{const.}, \tag{5}$$

where $C_1=\int_0^h f(\eta)d\eta=0.615$ is a shape factor determined from the similarity solution.

Integrating (1) from 0 to h and using (5) yields

$$w=u\frac{dh}{dr}\eta=u_s f(\eta)h'\eta \tag{6}$$

Writing (1) in the form

$$\frac{1}{2}\rho(u^2+w^2)\left(\frac{\partial(ru)}{\partial r}+\frac{\partial(rw)}{\partial z}\right)=0, \tag{7}$$

using (7) and the circular symmetry of the system allows the mechanical energy equation to be written as

$$\begin{aligned}\frac{\partial}{\partial r}(ur\frac{1}{2}\rho(u^2+w^2))+\frac{\partial}{\partial z}(wr\frac{1}{2}\rho(u^2+w^2))=\\ -ur\frac{dp}{dr}-\rho gur\frac{dh}{dr}+ru\mu(\frac{\partial^2 u}{\partial z^2})\end{aligned} \tag{8}$$

The mechanical energy equation (8) does not have a surface energy term as the differential element of fluid does not include the free surface. However, on integration of (8) from $z=0$ to the free surface $z=h$, we need to add the surface energy term. This is calculated as follows.

The rate of change of surface energy at a radial location r and in a differential volume $2\pi rh\Delta r$ is $2\pi\gamma r\frac{\Delta r}{\Delta t}$ and in the limit $\Delta r\to 0$

$$2\pi\gamma r\frac{\Delta r}{\Delta t}=2\pi\gamma r\bar{u}. \tag{9}$$

Assuming circular symmetry around the jet axis, we can write the rate of change of surface energy per unit angle as $\gamma r\bar{u}$. In the limit $\Delta r\to 0$, a balance on the flux of surface energy across the annular control volume (figure 2), yields

$$\lim_{\Delta r\to 0}\frac{(\gamma\bar{u}r)\Big|_{r+\Delta r}-(\gamma\bar{u}r)\Big|_r}{\Delta r}=\frac{d(\gamma\bar{u}r)}{dr}. \tag{10}$$

Substituting (6) in (8) and integrating the LHS of the equation w.r.t z and adding the surface energy term, yields

$$\begin{aligned}\int_0^h\frac{\partial}{\partial r}\left(\frac{\rho ru^3}{2}(1+(h'\eta)^2)dz\right)+\\ \int_0^h\frac{\partial}{\partial z}\left(\frac{\rho ru^3}{2}h'\eta(1+(h'\eta)^2)dz\right),\end{aligned} \tag{11}$$

$$\int_0^h\frac{\partial}{\partial r}\left(\frac{\rho ru^3}{2}(1+(h'\eta)^2)dz\right)+\frac{\rho ru_s^3}{2}h'\eta(1+h'^2). \tag{12}$$

Applying Leibniz's integral rule yields,

$$\begin{aligned}\frac{d}{dr}\int_0^h\left(\frac{\rho ru^3}{2}(1+(h'\eta)^2)\right)dz=\frac{d\left(\frac{\rho rhu^3}{2}\right)}{dr}\\ \int_0^1 f^3(\eta)(1+(h'\eta)^2)d\eta.\end{aligned} \tag{13}$$

Now using the fact that $u_s rh=$ constant, (13) can be written as

$$\begin{aligned}\frac{d}{dr}\int_0^h\left(\frac{\rho ru^3}{2}(1+(h'\eta)^2)\right)dz\\ =\rho u_s^2 hr\frac{du_s}{dr}\int_0^1 f^3(\eta)(1+(h'\eta)^2)d\eta.\end{aligned} \tag{14}$$

Similarly integrating the RHS of (8) w.r.t z and adding the surface energy term from (10) yields

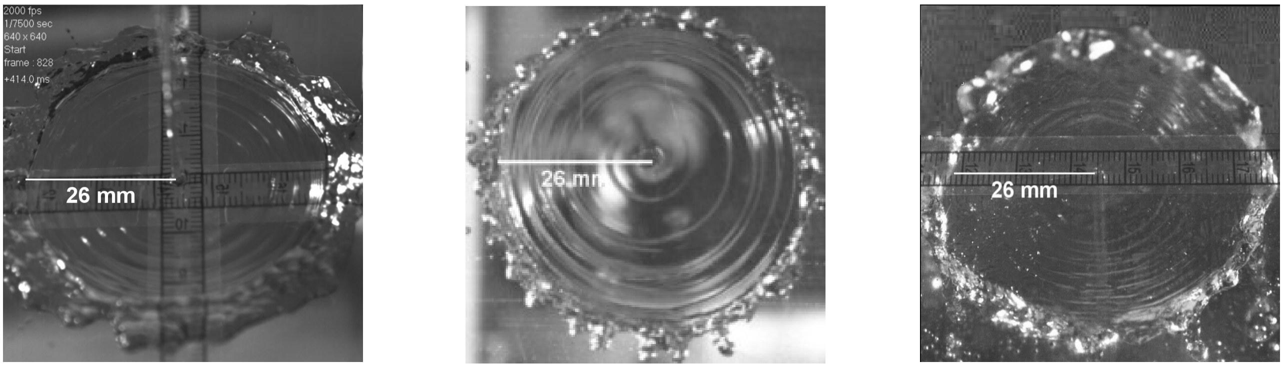
$$\begin{aligned}-\int_0^h ur\frac{dp}{dr}dz-\int_0^h\rho gur\frac{dh}{dr}dz+\int_0^h ru\mu(\frac{\partial^2 u}{\partial z^2})dz\\ +\frac{d(\gamma\bar{u}r)}{\partial r}.\end{aligned} \tag{15}$$

Then, using relation $u_s rh=$ constant implies $u_s r\frac{dh}{dr}=-h\frac{d(u_s r)}{dr}$, yields

$$\begin{aligned}-u_s rh\frac{dp}{dr}\int_0^1 f(\eta)d\eta-\rho gh^2\left(r\frac{du_s}{dr}+u_s\right)\int_0^1 f(\eta)d\eta+\\ \frac{\mu ru_s^2}{h}\int_0^1 f(\eta)f''(\eta)d\eta+\gamma\frac{d(\bar{u}r)}{dr}.\end{aligned} \tag{16}$$

Finally putting together the LHS of (14) and the RHS of (16) and using the relation $\bar{u}=C_1 u_s$ gives

$$\frac{du_s}{dr}=\frac{-u_s rhC_1\frac{dp}{dr}+C_1\gamma u_s+C_1\rho gu_s h^2+\frac{\mu ru_s^2}{h}\int_0^1 f(\eta)f''(\eta)d\eta}{(1-\frac{1}{We}-\frac{1}{Fr^2})\left(\rho u_s^2 hr\int_0^1 f^3(\eta)(1+(h'\eta)^2)d\eta\right)}. \tag{17}$$



(a) Horizontal surface; jet impinging from above

(b) Vertical surface; jet impinging horizontally

(c) Horizontal surface; jet impinging from below

Figure 1: Reproduced from Bhagat et al. (2018); Hydraulic jumps caused by a water jet impinging normally on surfaces with different orientations. In these three cases the jets are identical, produced from the same nozzle at the same flowrate, $Q = 1\,\mathsf{L\,min^{-1}}$, and the radius of the jump is observed to be independent of the orientation of the surface.

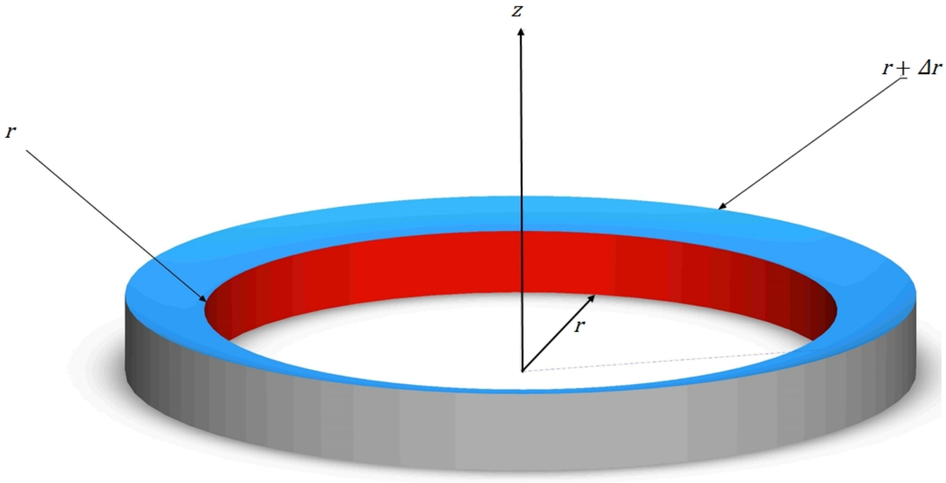


Figure 2: Control volume of the differential annular volume of the liquid

Here, the Weber number and Froude number are defined as, respectively,

$$We \equiv \frac{\rho u_s^2 h \int_0^1 f^3(\eta)(1+(h'\eta)^2)d\eta}{C_1\gamma}, \\ ,Fr \equiv \sqrt{\frac{u_s^2 \int_0^1 f^3(\eta)(1+(h'\eta)^2)d\eta}{C_1 g h}}. \tag{18}$$

Equation (17) was solved with the initial condition obtained from Watson's analysis of the growth of the boundary layer [3]. The boundary layer first occupies the whole film at r_{bl}, given by $\frac{r_{bl}}{d} = 0.1833\sqrt[3]{Re}$, where d is the nozzle diameter and the jet Reynolds number $Re = \frac{\rho U_o d}{\mu}$. At this location u_s is set equal to the mean jet velocity. Inside the jump radius h' remains very small therefore, to obtain an approximate solution upstream of the hydraulic jump, in (17) and (18), h' was set to zero. From Watson solution, we can calculate $\int_0^1 f(\eta)^3 d\eta = 0.4$ and $\int_0^1 f''(\eta)f(\eta)d\eta = -1.179$. Substituting the integrands in (17) and solving it with initial condition at $r = r_{bl}$, $u_s = U_o$ provides the subsequent radial values of the surface velocity. The location where $We^{-1} + Fr^{-2} = 1$, (17) becomes singular and there is a discontinuity in the film velocity and the liquid film thickness changes abruptly. Therefore, the condition for hydraulic jump is,

$$We^{-1} + Fr^{-2} = 1. \tag{19}$$

Results and discussion

Equation (19) shows that the hydraulic jump depends on the local We and Fr number. Bhagat et al.(2018) showed that in case of thin-film hydraulic jump Fr remains very high and the criterion is set by the Weber number [2]. They systematically changed the surface tension and the viscosity of the test liquids and showed that their theory gives excellent agreement with the experimental data. In the present work, we compare the theoretical prediction with the experimental data obtained for different surface orientations (data reproduced from [2]). Figure 3 compares the theoretical prediction obtained by the solution of (17), whereas the dashed (blue) line represents the theoretical prediction obtained by Bhagat et al.(2018). The two predictions overlap each other and show excellent agreement with the experimental data.

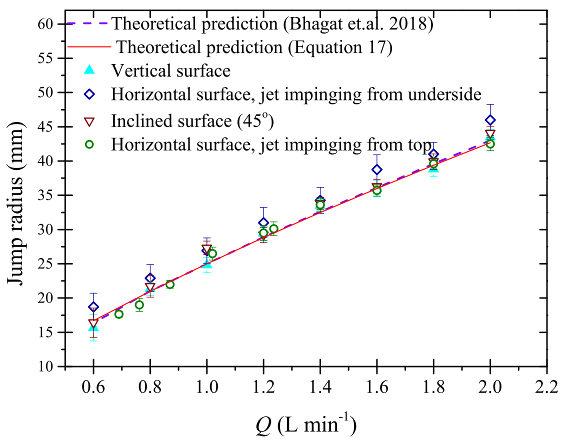


Figure 3: Comparison of theoretical prediction obtained using solution of (17) (red line) with the experimental results and obtained by Bhagat et al.(2018). The dashed line shows Bhagat et al.'s theoretical prediction.

Conclusions

The general theoretical treatment, including gravity and surface tension, presented in this paper show that the condition at a circular hydraulic jump is $We^{-1} + Fr^{-2} = 1$. In case of thin films, gravity does not play any significant role and the hydraulic jumps are created due to surface tension alone and the effective condition for a thin film hydraulic jump is $We \approx 1$. This calculation agrees with the conclusions drawn by Bhagat et al.(2018) and with both the flux average control volume analysis presented by Bhagat et al. and the differential analysis presented in this paper are in agreement.

Acknowledgements

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